Expanders and Morita-compatible exact crossed products

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An expander or expander family is a sequence of finite graphs $X_1, X_2, X_3, \ldots$ which is efficiently connected. A discrete group $G$ which contains an expander as a sub-graph of its Cayley graph is a counter-example to the Baum-Connes (BC) conjecture with coefficients. Some care must be taken with the definition of “contains”. It is sufficient that there exists a coarse embedding of an expander into the Cayley graph of $G$. M. Gromov outlined a method for constructing such a group. G. Arjantseva and T. Delzant completed the construction. Any group so obtained is known as a Gromov group (or Gromov monster). More recently examples have been constructed by Damian Osajda which truly have an expander as a sub-graph of the Cayley graph. These more recent examples and the Gromov groups are the only known examples of non-exact groups.
The left side of BC with coefficients “sees” any group as if the group were exact. This talk will indicate how to make a change in the right side of BC with coefficients so that the right side also “sees” any group as if the group were exact. This corrected form of BC with coefficients uses the unique minimal exact and Morita-compatible intermediate crossed product. For exact groups (i.e. all groups except the Gromov groups and the more recent Osajda examples) there is no change in BC with coefficients.
In the corrected form of BC with coefficients any Gromov group or Osajda group acting on the coefficient algebra obtained from its expander is not a counter-example.

Thus at the present time (January, 2015) there is no known counter-example to the corrected form of BC with coefficients.

The above is joint work with E. Guentner and R. Willett. This work is based on — and inspired by — a result of R. Willett and G. Yu, and is very closely connected to results in the thesis of M. Finn-Sell.
A discrete group $\Gamma$ which “contains” an expander in its Cayley graph is a counter-example to the usual (i.e. uncorrected) BC conjecture with coefficients.

Some care must be taken with the definition of “contains”.
An expander or expander family is a sequence of finite graphs

\[ X_1, X_2, X_3, \ldots \]

which is \textit{efficiently connected}. 
The isoperimetric constant $h(X)$

For a finite graph $X$ with vertex set $V$ and $|V|$ vertices

$$h(X) =: \min\{ \frac{|\partial F|}{|F|} \mid F \subset V \text{ and } |F| \leq \frac{|V|}{2} \}$$

$|F|$ = number of vertices in $F$. $F \subset V$.

$|\partial F|$ = number of edges in $X$ having one vertex in $F$ and one vertex in $V - F$. 

An **expander** is a sequence of finite graphs

\[ X_1, X_2, X_3, \ldots \]

such that

- Each \( X_j \) is connected.
- \( \exists \) a positive integer \( d \) such that all the \( X_j \) are \( d \)-regular.
- \( |X_n| \to \infty \) as \( n \to \infty \).
- \( \exists \) a positive real number \( \epsilon > 0 \) with
  \[ h(X_j) \geq \epsilon > 0 \quad \forall j = 1, 2, 3, \ldots \]
$G$ topological group

$G$ is assumed to be:
locally compact, Hausdorff, and second countable.

(second countable = The topology of $G$ has a countable base.)

**Examples**

<table>
<thead>
<tr>
<th>Category</th>
<th>Example</th>
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<tbody>
<tr>
<td>Lie groups</td>
<td>$\text{SL}(n, \mathbb{R})$</td>
</tr>
<tr>
<td>$p$-adic groups</td>
<td>$\text{SL}(n, \mathbb{Q}_p)$</td>
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<tr>
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Ordinary BC and BC with coefficients are for topological groups $G$ which are locally compact, Hausdorff, and second countable.

$EG$ denotes the universal example for proper actions of $G$.

EXAMPLE. If $\Gamma$ is a (countable) discrete group, then $E\Gamma$ can be taken to be the convex hull of $\Gamma$ within $l^2(\Gamma)$. 
$K^G_j(EG)$ denotes the Kasparov equivariant $K$-homology — with $G$-compact supports — of $EG$.

**Definition**

A closed subset $\Delta$ of $EG$ is $G$-compact if:

1. The action of $G$ on $EG$ preserves $\Delta$. and
2. The quotient space $\Delta/G$ (with the quotient space topology) is compact.
Definition

\[ K^G_j(EG) = \lim_{\Delta \subset EG \atop \Delta \text{ G-compact}} KK^j_G(C_0(\Delta), \mathbb{C}). \]

The direct limit is taken over all $G$-compact subsets $\Delta$ of $EG$.

$K^G_j(EG)$ is the Kasparov equivariant $K$-homology of $EG$ with $G$-compact supports.
Ordinary BC

Conjecture

For any $G$ which is locally compact, Hausdorff and second countable

$$K_j^G(EG) \rightarrow K_j(C_r^*G) \quad j = 0, 1$$

is an isomorphism
Corollaries of BC

Novikov conjecture $=$ homotopy invariance of higher signatures
Stable Gromov Lawson Rosenberg conjecture (Hanke + Schick)

Idempotent conjecture
Kadison Kaplansky conjecture
Mackey analogy (Higson)
Exhaustion of the discrete series via Dirac induction
(Parthasarathy, Atiyah + Schmid, V. Lafforgue)
Homotopy invariance of $\rho$-invariants
(Keswani, Piazza + Schick)
$G$ topological group
locally compact, Hausdorff, second countable

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Let $A$ be a $G - C^*$ algebra i.e. a $C^*$ algebra with a given continuous action of $G$ by automorphisms.

$$G \times A \rightarrow A$$

BC with coefficients

**Conjecture**

For any $G$ which is locally compact, Hausdorff, and second countable and any $G - C^*$ algebra $A$

$$K^G_j(EG, A) \rightarrow K^j(C^*_r(G, A)) \quad j = 0, 1$$

is an isomorphism.
Definition

\[ K^G_j(EG, A) = \lim_{\Delta \subset EG, \Delta \text{ G-compact}} KK^j_G(C_0(\Delta), A). \]

The direct limit is taken over all \( G \)-compact subsets \( \Delta \) of \( EG \).

\( K^G_j(EG, A) \) is the Kasparov equivariant \( K \)-homology of \( EG \) with \( G \)-compact supports and with coefficient algebra \( A \).
THEOREM [N. Higson + G. Kasparov] Let $\Gamma$ be a discrete (countable) group which is amenable or a-t-menable, and let $A$ be any $\Gamma - C^*$ algebra. Then

$$\mu: K_j^\Gamma(ET, A) \to K_jC_r^*(\Gamma, A)$$

is an isomorphism. $j = 0, 1$
THEOREM [V. Lafforgue] Let $\Gamma$ be a discrete (countable) group which is hyperbolic (in Gromov’s sense), and let $A$ be any $\Gamma - C^*$ algebra. Then

$$\mu: K_j^\Gamma(EG, A) \rightarrow K_j C_r^*(\Gamma, A)$$

is an isomorphism. $j = 0, 1$
$SL(3, \mathbb{Z})$
Basic property of \( C^* \) algebra K-theory

SIX TERM EXACT SEQUENCE

Let

\[
0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0
\]

be a short exact sequence of \( C^* \) algebras.

Then there is a six term exact sequence of abelian groups

\[
\begin{array}{c}
K_0 I & \longrightarrow & K_0 A & \longrightarrow & K_0 B \\
\uparrow & & \uparrow & & \uparrow \\
K_1 B & \leftarrow & K_1 A & \leftarrow & K_1 I
\end{array}
\]
DEFINITION. \( G \) is exact if whenever

\[
0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0
\]

is an exact sequence of \( G - C^* \) algebras, then

\[
0 \rightarrow C^*_r(G, I) \rightarrow C^*_r(G, A) \rightarrow C^*_r(G, B) \rightarrow 0
\]

is an exact sequence of \( C^* \) algebras.
LEMMA. Let $$0 \to I \to A \to B \to 0$$ be an exact sequence of $G - C^*$ algebras. Assume that $G$ is exact. Then there is a six term exact sequence of abelian groups

$$K_0 C_r^*(G, I) \to K_0 C_r^*(G, A) \to K_0 C_r^*(G, B)$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\quad K_1 C_r^*(G, B) \quad \quad K_1 C_r^*(G, A) \quad \quad K_1 C_r^*(G, I)$$
The left side of BC with coefficients “sees” any $G$ as if $G$ were exact.
LEMMA. For any locally compact Hausdorff second countable topological group $G$ and any exact sequence

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

of $G - C^*$ algebras, there is a six term exact sequence of abelian groups

There is no hypothesis here that $G$ is exact.

$$
\begin{array}{c}
K_0^G(\mathcal{E}G, I) \rightarrow K_0^G(\mathcal{E}G, A) \rightarrow K_0^G(\mathcal{E}G, B) \\
\uparrow & & \uparrow \\
K_1^G(\mathcal{E}G, B) & \leftarrow & K_1^G(\mathcal{E}G, A) & \leftarrow & K_1^G(\mathcal{E}G, I)
\end{array}
$$
QUESTION. Do non-exact groups exist?

ANSWER. If a discrete group $\Gamma$ “contains” an expander in its Cayley graph, then $\Gamma$ is not exact.

“contains” = There exists an expander $X$ and a map

$$f : X \longrightarrow \text{Cayley graph}(\Gamma)$$

such that $f$ is a uniform embedding in the sense of coarse geometry of metric spaces.
If $\Gamma$, “contains” an expander in its Cayley graph, then there exists an exact sequence

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

of $\Gamma - C^*$ algebras, such that

$$K_0 C^*_r(\Gamma, I) \rightarrow K_0 C^*_r(\Gamma, A) \rightarrow K_0 C^*_r(\Gamma, B)$$

is not exact.

Since

$$K_0^\Gamma (\overline{E \Gamma}, I) \rightarrow K_0^\Gamma (\overline{E \Gamma}, A) \rightarrow K_0^\Gamma (\overline{E \Gamma}, B)$$

is exact, such a $\Gamma$ is a counter-example to BC with coefficients.
For the construction (given such a $\Gamma$) of the relevant exact sequence

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

of $\Gamma - C^*$ algebras,
see the paper of N.Higson and V. Lafforgue and G. Skandalis. $A$ is (the closure of) the sub-algebra of $L^\infty(\Gamma)$ consisting of functions which are supported in $R$-neighborhoods of the expander.

Also, see the thesis of M. Finn-Sell.
Theorem (N. Higson and G. Kasparov)

If $\Gamma$ is a discrete group which is amenable (or a-t-menable), then BC with coefficients is true for $\Gamma$.

Theorem (V. Lafforgue)

If $\Gamma$ is a discrete group which is hyperbolic (in Gromov’s sense), then BC with coefficients is true for $\Gamma$. 
Possible Happy Ending

A possible happy ending is:
If $G$ is exact, then BC with coefficients is true for $G$.

PROBLEM. Is BC (i.e. ordinary BC = BC without coefficients) true for $SL(3,\mathbb{Z})$?
STOP!!!! HOLD EVERYTHING!!!!
Consider the result of Rufus Willett and Guoliang Yu:

**Theorem**

Let $\Gamma$ be the Gromov group and let $A$ be the $\Gamma$ - $C^*$ algebra obtained by mapping an expander to the Cayley graph of $\Gamma$. Then

$$K_j^\Gamma(EG, A) \rightarrow K_j(C^*_{max}(\Gamma, A)) \quad j = 0, 1$$

is an isomorphism.

This theorem indicates that for non-exact groups the right side of BC with coefficients has to be reformulated.
For exact groups (i.e. all groups except the Gromov groups) no change should be made in the statement of BC with coefficients.
With $G$ fixed, $\{G - C^* \text{ algebras}\}$ denotes the category whose objects are all the $G - C^*$ algebras.

Morphisms in $\{G - C^* \text{ algebras}\}$ are $*$-homomorphisms which are $G$-equivariant.

$\{C^* \text{ algebras}\}$ denotes the category of $C^*$ algebras. Morphisms in $\{C^* \text{ algebras}\}$ are $*$-homomorphisms.
A crossed-product is a functor, denoted $A \mapsto C^*_\tau(G, A)$ from \{\text{$G$ – $C^*$ algebras}\} to \{\text{$C^*$ algebras}\}

$$C^*_\tau: \{\text{$G$ – $C^*$ algebras}\} \longrightarrow \{\text{$C^*$ algebras}\}$$
“intermediate” = “between the max and the reduced crossed-product”

For an intermediate crossed-product $C_\tau^*$ there are surjections:

$$C_{max}^*(G, A) \rightarrow C_\tau^*(G, A) \rightarrow C_r^*(G, A)$$

Denote by $\tau(G, A)$ the kernel of the surjection $C_{max}^*(G, A) \rightarrow C_\tau^*(G, A)$

$$0 \rightarrow \tau(G, A) \rightarrow C_{max}^*(G, A) \rightarrow C_\tau^*(G, A) \rightarrow 0$$

is exact.
Denote by $\epsilon(G, A)$ the kernel of $C_{\text{max}}^*(G, A) \longrightarrow C_r^*(G, A)$.

$$0 \longrightarrow \epsilon(G, A) \longrightarrow C_{\text{max}}^*(G, A) \longrightarrow C_r^*(G, A) \longrightarrow 0$$

is exact.

An intermediate crossed-product $C_\tau^*$ is then a function $\tau$ which assigns to each $G - C^*$ algebra $A$ a norm closed ideal $\tau(G, A)$ in $C_{\text{max}}^*(G, A)$ such that:
For each $G - C^*$ algebra $A$, $\tau(G, A) \subseteq \epsilon(G, A)$.

For each morphism $A \to B$ in $\{G - C^*$ algebras\}
the resulting $*$-homomorphism $C^*_{max}(G, A) \to C^*_{max}(G, B)$
maps $\tau(G, A)$ to $\tau(G, B)$.

$$C^*_\tau(G, A) = C^*_{max}(G, A) / \tau(G, A)$$
An intermediate crossed-product $C^*_{\tau}$ is exact if whenever

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is an exact sequence in $\{G - C^*\text{ algebras}\}$ the resulting sequence in $\{C^*\text{ algebras}\}$

$$0 \rightarrow C^*_{\tau}(G, A) \rightarrow C^*_{\tau}(G, B) \rightarrow C^*_{\tau}(G, C) \rightarrow 0$$

is exact.
Equivalently:
An intermediate crossed-product $C^*_\tau$ is exact if whenever

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is an exact sequence in \{G - C^* algebras\} the resulting sequence in \{C^* algebras\}

$$0 \rightarrow \tau(G, A) \rightarrow \tau(G, B) \rightarrow \tau(G, C) \rightarrow 0$$

is exact.
Set $H_G = L^2(G) \oplus L^2(G) \oplus \ldots$ \quad $\mathcal{K}_G = \mathcal{K}(H_G)$

An intermediate crossed product $C^*_\tau$ is Morita-compatible if for any $G - C^*$ algebra $A$ the natural isomorphism of $C^*$ algebras

$$C^*_{max}(G, A \otimes \mathcal{K}_G) = C^*_{max}(G, A) \otimes \mathcal{K}_G$$

descends to give an isomorphism of $C^*$ algebras

$$C^*_\tau(G, A \otimes \mathcal{K}_G) = C^*_\tau(G, A) \otimes \mathcal{K}_G$$
QUESTION. Given $G$, does there exist a unique minimal intermediate crossed product which is exact and Morita-compatible?

PROPOSITION. (E. Kirchberg, P.Baum& E.Guentner& R.Willett) For any locally compact Hausdorff second countable topological group $G$ there exists a unique minimal intermediate crossed product which is exact and Morita-compatible.

Denote the unique minimal intermediate exact and Morita-compatible crossed-product by $C^*_{exact}$. 
Reformulation of BC with coefficients.

CONJECTURE. Let $G$ be a locally compact Hausdorff second countable topological group, and let $A$ be a $G - C^*$ algebra, then

$$K^G_j(EG, A) \longrightarrow K_j(C^*_e(G, A))$$

$j = 0, 1$

is an isomorphism.
Theorem (PB and E. Guentner and R. Willett)

Let $\Gamma$ be a Gromov group or an Osajda group and let $A$ be the $\Gamma$ - $C^*$ algebra obtained by mapping an expander to the Cayley graph of $\Gamma$. Then

$$K_j^\Gamma(\mathcal{E}\Gamma, A) \to K_j(C^*_{exact}(\Gamma, A)) \quad j = 0, 1$$

is an isomorphism.
Implications of the corrected conjecture

Novikov (homotopy invariance of higher signatures) and stable Gromov-Lawson-Rosenberg are implied by the corrected conjecture. ✓

Kadison-Kaplansky : If $\Gamma$ is a torsion-free discrete group, then in $C^*_r \Gamma$ there are no idempotent elements (other than 0 and 1).

Kadison-Kaplansky is not implied by the corrected conjecture.
Implied by validity of the corrected conjecture:

If \( \Gamma \) is a torsion-free discrete group, then in the Banach algebra \( l^1\Gamma \) there are no idempotent elements (other than 0 and 1).

\[ l^1\Gamma \subset C^*_r\Gamma \]

**Question.** Is it possible for a torsion-free discrete group \( \Gamma \) to have no idempotent elements (other than 0 and 1) in \( l^1\Gamma \) — and to have idempotent elements (other than 0 and 1) in \( C^*_r\Gamma \)?
Revised Kadison-Kaplansky Conjecture. If \( \Gamma \) is a torsion-free discrete group which is exact, then in \( C^*_r \Gamma \) there are no idempotent elements (other than 0 and 1).
\[ F\Gamma := \left\{ \sum_{\gamma \in \Gamma} \lambda_{\gamma}[\gamma] \mid \text{order} \,(\gamma) < \infty \right\} \]

**Finite Formal Sums**

\( F\Gamma \) is a vector space over \( \mathbb{C} \)

\[
\left( \sum_{\gamma \in \Gamma} \lambda_{\gamma}[\gamma] \right) + \left( \sum_{\gamma \in \Gamma} \mu_{\gamma}[\gamma] \right) = \sum_{\gamma \in \Gamma} (\lambda_{\gamma} + \mu_{\gamma})[\gamma]
\]

\[
\lambda \left( \sum_{\gamma \in \Gamma} \lambda_{\gamma}[\gamma] \right) = \sum_{\gamma \in \Gamma} \lambda \lambda_{\gamma}[\gamma] \quad \lambda \in \mathbb{C}
\]
\( F\Gamma \) is a \( \Gamma \)-module

\[ \Gamma \times F\Gamma \to F\Gamma \]

\[ g \in \Gamma \quad \sum_{\gamma \in \Gamma} \lambda_\gamma [\gamma] \in F\Gamma \]

\[ g \left( \sum_{\gamma \in \Gamma} \lambda_\gamma [\gamma] \right) = \sum_{\gamma \in \Gamma} \lambda_\gamma [g \gamma g^{-1}] \]
\[ H_j(\Gamma; F\Gamma) := \]

the \( j \)-th homology group of \( \Gamma \) with coefficients the \( \Gamma \)-module \( F\Gamma \)

\[ j = 0, 1, 2, \ldots \]

**Remark**

This is standard homological algebra, and is pure algebra (i.e. \( \Gamma \) is a discrete group and \( F\Gamma \) is a non-topologized module over \( \Gamma \)).
\[ \text{ch : } K_{*}^{\text{top}}(\Gamma) \rightarrow H_{*}(\Gamma; F\Gamma) \]

\[ \text{ch : } K_{j}^{\text{top}}(\Gamma) \rightarrow \bigoplus_{\ell} H_{j+2\ell}(\Gamma; F\Gamma) \]

\[ j = 0, 1 \]

**Proposition**

\[ K_{j}^{\text{top}}(\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow \bigoplus_{\ell} H_{j+2\ell}(\Gamma; F\Gamma) \]

is an isomorphism of vector spaces over \( \mathbb{C} \).
$E\Gamma$ denotes the universal example for proper actions of $\Gamma$. $E\Gamma$ can be taken to be the convex hull of $\Gamma$ within $l^2(\Gamma)$. 
Example

Give $\Gamma$ the measure in which each $\gamma \in \Gamma$ has mass one. Consider the Hilbert space $l^2(\Gamma)$. $\Gamma$ acts on $l^2(\Gamma)$ via the (left) regular representation of $\Gamma$. $\Gamma$ embeds into $l^2(\Gamma)$ via $\Gamma \hookrightarrow l^2(\Gamma)$. $\gamma \in \Gamma$, $\gamma \mapsto [\gamma]$ where $[\gamma]$ is the Dirac function at $\gamma$. Within $l^2(\Gamma)$ let Convex-Hull($\Gamma$) be the smallest convex set which contains $\Gamma$. The points of Convex-Hull($\Gamma$) are all the finite sums

$$t_0[\gamma_0] + t_1[\gamma_1] + \cdots + t_n[\gamma_n]$$

with $t_j \in [0, 1]$ for $j = 0, 1, \ldots, n$ and $t_0 + t_1 + \cdots + t_n = 1$

The action of $\Gamma$ on $l^2(\Gamma)$ preserves Convex-Hull($\Gamma$).

$\Gamma \times \text{Convex-Hull}(\Gamma) \longrightarrow \text{Convex-Hull}(\Gamma)$

$E\Gamma$ can be taken to be Convex-Hull($\Gamma$) with this action of $\Gamma$. 
$X$ topological space

$\Gamma \times X \to X$ continuous action of $\Gamma$ on $X$

$\tilde{X} := \{(\gamma, x) \in \Gamma \times X \mid \gamma x = x\}$

$\tilde{X} \subset \Gamma \times X$

$\Gamma \times \tilde{X} \to \tilde{X}$

$g(\gamma, x) = (g\gamma g^{-1}, gx) \quad g \in \Gamma, \ (\gamma, x) \in \tilde{X}$

**Lemma**

$$H_j(\Gamma; F\Gamma) = H_j(\tilde{E}\Gamma / \Gamma; \mathbb{C})$$

$j = 0, 1, 2, \ldots$
\[ K^\text{top}_*(\Gamma) := \{(M, E)\}/ \sim \]
\[ \text{ch} : K^\text{top}_*(\Gamma) \to H_*(\Gamma; F\Gamma) \]
\[ \text{ch}(M, E) := \tilde{\epsilon}_*(\text{ch}_\Gamma(E) \cap \text{td}_\Gamma(M)) \]
\[ \text{ch}_\Gamma(E) \cap \text{td}_\Gamma(M) \in H_*(\tilde{M}/\Gamma; \mathbb{C}) \]

\[ \text{ch}_\Gamma(E) \cap \text{td}_\Gamma(M) \] is the Atiyah-Singer formula for \( \text{Index}_\Gamma(D_E) \)
(\(M, E\))

Action of \(\Gamma\) on \(M\) is proper

\[\exists \text{ a continuous } \Gamma\text{-equivariant map } \epsilon : M \to \overline{E\Gamma}\]

\[\tilde{\epsilon} : \tilde{M} \to \overline{E\Gamma}\]

\[\tilde{\epsilon} : \tilde{M}/\Gamma \to \overline{E\Gamma}/\Gamma\]

\[\tilde{\epsilon}_* : H_*(\tilde{M}/\Gamma; \mathbb{C}) \to H_*(\overline{E\Gamma}/\Gamma; \mathbb{C}) = H_*(\Gamma; F\Gamma)\]

\[\text{ch} : K_{\text{top}}^*(\Gamma) \to H_*(\Gamma; F\Gamma)\]

\[\text{ch}(M, E) = \tilde{\epsilon}_*(\text{ch}_\Gamma(E) \cap \text{td}_\Gamma(M))\]
$\text{ch}_\Gamma(E) \cap \text{td}_\Gamma(M) \in H_*(\tilde{M}/\Gamma; \mathbb{C})$

How is $\text{ch}_\Gamma(E) \cap \text{td}_\Gamma(M)$ defined?

**Two methods:**

(1) Spectral triple + cyclic cohomology

(2) Classical algebraic topology
\[ \text{ch} : K^\text{top}_*(\Gamma) \to H_*(\Gamma; F\Gamma) \]

\[ \text{ch} : K^\text{top}_j(\Gamma) \to \bigoplus_{\ell} H_{j+2\ell}(\Gamma; F\Gamma) \]

\[ j = 0, 1 \]

**Proposition**

\[ K^\text{top}_j(\Gamma) \otimes \mathbb{Z} \mathbb{C} \to \bigoplus_{\ell} H_{j+2\ell}(\Gamma; F\Gamma) \]

is an isomorphism of vector spaces over \( \mathbb{C} \).
$K^j_j(E\Gamma)$ denotes the Kasparov equivariant $K$-homology — with $\Gamma$-compact supports — of $E\Gamma$.

**Definition**

A closed subset $\Delta$ of $E\Gamma$ is $\Gamma$-compact if:

1. The action of $\Gamma$ on $E\Gamma$ preserves $\Delta$.
   and
2. The quotient space $\Delta/\Gamma$ (with the quotient space topology) is compact.
Definition

\[ K^\Gamma_j(E\Gamma) = \lim_{\Delta \subset E\Gamma, \Delta \text{ \Gamma-compact}} KK^j_\Gamma(C_0(\Delta), \mathbb{C}). \]

The direct limit is taken over all \( \Gamma \)-compact subsets \( \Delta \) of \( E\Gamma \).

\( K^\Gamma_j(E\Gamma) \) is the Kasparov equivariant \( K \)-homology of \( E\Gamma \) with \( \Gamma \)-compact supports.
\[ \tau : K^\text{top}_j(\Gamma) \to K^\Gamma_j(\overline{E\Gamma}) \]
\[(M, E) \mapsto \epsilon_\ast [D_E] \]

where

\[ \epsilon : M \longrightarrow \overline{E\Gamma} \] is (as above) a continuous \( \Gamma \)-equivariant map and

\[ [D_E] \in KK^j_\Gamma(C_0(M), \mathbb{C}) \] is the element in the Kasparov equivariant \( K \)-homology of \( M \) determined by \( D_E \).

\[ j = 0, 1 \]
Theorem (P.B. + N. Higson + T. Schick)

\[ \tau : K_{j}^{\text{top}}(\Gamma) \rightarrow K_{j}^{\Gamma}(E \Gamma) \]

is an isomorphism \( j = 0, 1 \)
Gromov’s principle

There is no statement about all finitely presentable discrete groups which is both non-trivial and true.
Theorem

\[ K^\top_j(\Gamma) \xrightarrow{\sim} K^\Gamma_j(E\Gamma) \]

\[ \mathbb{C} \otimes_{\mathbb{Z}} K^\top_j(\Gamma) \xrightarrow{\sim} \bigoplus_{\ell} H_{j+2\ell}(\Gamma; F\Gamma) \]

\[ j = 0, 1 \]

Question. Does this theorem violate Gromov’s principle?
Theorem (PB and R. Willett)

Let $\Gamma$ be a Gromov group and let $A$ be the $\Gamma - C^*$ algebra obtained by mapping an expander to the Cayley graph of $\Gamma$. Then

$$K^\Gamma_j (\underline{E\Gamma}, A) \to K_j (C^*_\text{exact}(\Gamma, A)) \quad j = 0, 1$$

is an isomorphism.
Talks given using this file:
January 7, 2011, Joint Mathematics Meetings, New Orleans, Special Session on Expanders
January 4, 2012, Joint Mathematics Meetings, Boston, Special Session on Generalized Cohomology Theories in Engineering Practice
February 27, 2012, IMPAN Non-commutative geometry seminar
March 8, 2012, University of Warsaw, Cathedra Mathematical Methods in Physics
September 8, 2012, University of New Brunswick, Non-Commutative Geometry Workshop
November 7, 2012, University of Hawaii, Non-Commutative Geometry Seminar
November 15, 2012, Tohoku University, Geometry Seminar
December 14, 2012, Australian National University, Canberra, Baum Fest
January 16, 2013, University of Nijmegen, Mathematical Physics Seminar
May 4, 2013, NCGOA, Vanderbilt University, Nashville
June 25, 2013, Fields Institute, Marc Rieffel 75, Toronto